

# Discontinuous integral control for mechanical systems

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## Abstract

For mechanical systems we present a controller able to track an unknown smooth signal, converging in finite time and by means of a continuous control signal. The control scheme is insensitive against unknown perturbations with bounded derivative. The controller consists of a non locally Lipschitz state feedback control law, and a discontinuous integral controller, that is able to estimate the unknown perturbation and to compensate for it. To complete an output feedback control a continuous observer for the velocity is added. It is shown that the closed loop consisting of state feedback, state observer and discontinuous integral controller has an equilibrium point that is globally, finite time stable, despite of perturbations with bounded derivative. The proof is based on a new smooth Lyapunov function.

## I. INTRODUCTION

We consider in this paper a second order system

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= f(\xi_1, \xi_2, t) + \rho(t) + \tau\end{aligned}\quad (1)$$

where  $\xi_1 \in \mathbb{R}$  and  $\xi_2 \in \mathbb{R}$  are the states,  $\tau \in \mathbb{R}$  is the control variable,  $f(\xi_1, \xi_2, t)$  is some known function, while the term  $\rho(t)$  corresponds to uncertainties and/or perturbations. System 1 can represent a mechanical system, where  $\xi_1$  is the position and  $\xi_2$  is the velocity. An important control task is to track a smooth time varying reference  $r(t)$ , i.e. if one defines the tracking error  $z_1 = \xi_1 - r$  and  $z_2 = \xi_2 - \dot{r}$  the objective is to asymptotically stabilize the origin of system

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(\xi_1, \xi_2, t) + \rho(t) - \ddot{r}(t) + \tau.\end{aligned}\quad (2)$$

With the control  $\tau = u - f(\xi_1, \xi_2, t) + \ddot{r}(t)$  the system becomes

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \rho(t),\end{aligned}\quad (3)$$

where the perturbation  $\rho(t)$  is a time varying signal, not vanishing at the origin (i.e. when  $x = 0$  the perturbation can still be acting). We notice that it is possible not to feed the second derivative of the reference  $\ddot{r}(t)$  to the control  $\tau$ . In this case it will be considered as part of the perturbation term  $\rho(t)$ .

Under the stated hypothesis it is well known that a continuous, memoryless state feedback  $u = k(x)$  is not able to stabilize  $x = 0$ . This is so, because the controller has to satisfy with the condition  $k(0) = 0$ , since the closed loop has to have an equilibrium at the origin for vanishing perturbation. But if the perturbation does not vanish, then the origin cannot be anymore an equilibrium point. Discontinuous controllers, as the first order Sliding Mode (SM) ones [6], [5] are able to solve the problem for non vanishing (or persistently acting) bounded perturbations. However, they require the design of a sliding surface that is reached in finite time, but the target  $x = 0$  is attained only asymptotically fast, and at the cost of a high frequency switching of the control signal (the so called *chattering*), that has a negative effect in the actuator, and excites unmodelled dynamics of the plant. Higher Order Sliding Modes (HOSM) [7], [14], [13], [11], [17] provide a discontinuous controller for systems of relative degree higher than one to robustly stabilize the origin  $x = 0$  despite of bounded perturbations, but again at the expense of chattering. A natural alternative consists in adding an integrator, i.e. defining a new state  $z = u + \rho(t)$ , with  $\dot{z} = v$  and designing a third order HOSM controller for the new control variable  $v$ . This allows to reach the origin in finite time, and it will be insensitive to Lipschitz perturbations, i.e. with  $\dot{\rho}(t)$  bounded. In this form a continuous control signal  $u$  will be obtained, so that the chattering effect is reduced. However, this requires feedback not only the two states  $x_1$  and  $x_2$  but also the state  $z$ , which is unknown due to the unknown perturbation. Moreover, to implement an output feedback controller (assuming that only the position  $x_1$  is measured) it is necessary to differentiate two times the position  $x_1$ , with the consequent noise amplification effect.

In the case of (almost) constant perturbations  $\rho(t)$  a classical solution to the robust regulation problem is the use of integral action, as for example in the PID control[1]. The linear solution would consist of a state feedback plus an integral action,  $u = -k_1 x_1 - k_2 x_2 + z$ ,  $\dot{z} = -k_3 x_1$ . This controller requires only to feedback the position and the velocity. For an output feedback it would be only necessary to estimate the velocity (with the D action for example). In contrast to the HOSM controller this PID control is only able to reject constant perturbations, instead of Lipschitz ones, and it will reach the target

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only exponentially fast, and not in finite time. By the Internal Model Principle it would be possible to reject exactly any kind of time varying perturbations  $\rho(t)$ , for which a dynamical model (so called an exosystem) is available. However this would increase the complexity (order) of the controller, since this exosystem has to be included in the control law.

Here we provide a solution to the problem, that is somehow an intermediate solution between HOSM and PID control. Similar to the HOSM control our solution uses a discontinuous integral action, it can compensate perturbations with bounded derivative ( $\rho(t)$  is Lipschitz) and the origin is reached in finite time. So it can solve not only regulation problems (where  $\rho$  is constant) but also tracking problems (with  $\rho$  time varying) in finite time and with the same complexity of the controller. Similar to the PID control the proposed controller provides a continuous control signal (avoiding chattering) and it requires only to feedback position and velocity. We also provide for a (non classical) D term, i.e. a finite time converging observer, to estimate the velocity. This basic idea has been already presented in our previous work [20]. In the present one we give a much simpler Lyapunov-based proof, and we also include an observer in the closed loop together with its Lyapunov proof. Our solution can be seen as a generalization of the Super Twisting control for systems of relative degree one [17], [7], [14], [11], [15] to systems with relative degree two.

The rest of the paper is organized as follows. In next Section II we present the main result: The Discontinuous Integral Controller with and without observer and give some discussion on the algorithms. Section III is dedicated to present the Lyapunov-based proof of the convergence of the closed loop for the proposed control algorithms. In Section IV we give an illustrative example with some simulations and in Section V some conclusions are drawn.

*Notation 1:* To simplify the presentation we introduce the following notation. For a real variable  $z \in \mathbb{R}$  and a real number  $p \in \mathbb{R}$  the symbol  $\lfloor z \rfloor^p = |z|^p \text{sign}(z)$ , i.e. the signed power  $p$  of  $z$ . Note that  $\lfloor z \rfloor^2 = |z|^2 \text{sign}(z) \neq z^2$ , and if  $p$  is an odd number then  $\lfloor z \rfloor^p = z^p$ . Note also in particular, that

$$\begin{aligned} \lfloor z \rfloor^0 &= \text{sign}(z), & \lfloor z \rfloor^0 z^p &= |z|^p \\ \lfloor z \rfloor^0 |z|^p &= \lfloor z \rfloor^p, & \lfloor z \rfloor^p \lfloor z \rfloor^q &= \lfloor z \rfloor^{p+q} \end{aligned}$$

## II. DISCONTINUOUS INTEGRAL CONTROLLER

For the robust finite time stabilization of the origin of system (3) we propose a nonlinear, homogeneous state feedback control law, which is able to stabilize the origin in finite time in the absence of non vanishing perturbations, and a discontinuous integral controller is added to compensate for the persistently acting perturbations. In contrast to the continuous integral controller, that can only compensate for (almost) constant perturbations, the discontinuous one can deal with time varying perturbations which are Lipschitz continuous, that is, their derivatives exist almost everywhere and it is uniformly bounded. The control signal of the controller is continuous, so that the chattering effect of the SM and HOSM controllers is avoided.

*Theorem 1:* Consider the plant (3) with Lipschitz continuous perturbation signal  $\rho(t)$  with Lipschitz constant  $L$ . Then the control law

$$\begin{aligned} u &= -k_1 \lfloor x_1 \rfloor^{\frac{1}{3}} - k_2 \lfloor x_2 \rfloor^{\frac{1}{2}} + z \\ \dot{z} &= -k_3 \left[ x_1 + k_4 x_2^{\frac{3}{2}} \right]^0 \end{aligned} \quad (4)$$

can stabilize the origin in finite time for any  $k_4$  and appropriate designed gains  $k_1, k_2, k_3$ . ■

This Theorem shows that with the addition of the discontinuous integral term it is possible to eliminate completely the effect of the Lipschitz perturbation  $\delta(t)$ , that cannot be (fully) compensated by the state feedback  $u = -k_1 \lfloor x_1 \rfloor^{\frac{1}{3}} - k_2 \lfloor x_2 \rfloor^{\frac{1}{2}}$  alone. In fact, the integral controller can be interpreted as a perturbation estimator, since  $z(t) = -\delta(t)$  after a finite time.

It is remarkable, that the observer itself has not been designed to be robust against the perturbation, but the discontinuous integral controller is able to compensate for it. Note also that the input to the discontinuous integrator  $y = x_1 + k_4 x_2^{\frac{3}{2}}$  can be a combination of the position (with relative degree two) and the velocity (with a relative degree one). The value of  $k_4$  can be arbitrary (including zero), so that the velocity is not necessary for the integral action. For  $k_4 > 0$  this output can be seen as a passive output of the system  $(x_1, x_2)$ . However, it is necessary to have the position in this signal, otherwise the closed loop will be unstable.

By performing a linear transformation of the plant (3) with controller (4)  $\xi = \lambda x$ , for any some  $\lambda > 0$ , it is easy to show that if the gains  $(k_1, k_2, k_3, k_4)$  achieve the objective for a perturbation with Lipschitz constant  $L$ , then the gains  $(\lambda^{\frac{2}{3}} k_1, \lambda^{\frac{1}{2}} k_2, \lambda k_3, \lambda^{-\frac{3}{2}} k_4)$  will also stabilize the system for a perturbation with Lipschitz constant  $\lambda L$ .

The implementation of controller (4) requires the measurement of both states  $x_1, x_2$ . If only the position is measured a finite time convergent observer for  $x_2$  can be implemented, so that an output feedback control is obtained.

*Theorem 2:* Consider the plant (3) with Lipschitz continuous perturbation signal  $\rho(t)$  with Lipschitz constant  $L$ . Then the output feedback control law

$$\begin{aligned} \dot{\hat{x}}_1 &= -l_1 \lfloor \hat{x}_1 - x_1 \rfloor^{\frac{2}{3}} + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -l_2 \lfloor \hat{x}_1 - x_1 \rfloor^{\frac{1}{3}} - k_1 \lfloor x_1 \rfloor^{\frac{1}{3}} - k_2 \lfloor \hat{x}_2 \rfloor^{\frac{1}{2}} \\ u &= -k_1 \lfloor x_1 \rfloor^{\frac{1}{3}} - k_2 \lfloor \hat{x}_2 \rfloor^{\frac{1}{2}} + z \\ \dot{z} &= -k_3 \left[ x_1 + k_4 \hat{x}_2^{\frac{3}{2}} \right]^0, \end{aligned} \quad (5)$$

can stabilize the origin in finite time for appropriate designed gains  $k_1, k_2, k_3, k_4, l_1$  and  $l_2$ .  $\blacksquare$

Similarly to the previous case it follows that if the gains  $(k_1, k_2, k_3, k_4, l_1, l_2)$  achieve the objective for a perturbation with Lipschitz constant  $L$ , then the gains  $(\lambda^{\frac{2}{3}}k_1, \lambda^{\frac{1}{2}}k_2, \lambda k_3, \lambda^{-\frac{3}{2}}k_4, \lambda^{\frac{1}{3}}l_1, \lambda^{\frac{2}{3}}l_2)$  will also stabilize the system for a perturbation with Lipschitz constant  $\lambda L$ . System (3) with controller (5) is given by the dynamics

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 [x_1]^{\frac{1}{3}} - k_2 [\hat{x}_2]^{\frac{1}{2}} + z + \rho(t), \\ \dot{\hat{x}}_1 &= -l_1 [\hat{x}_1 - x_1]^{\frac{2}{3}} + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -l_2 [\hat{x}_1 - x_1]^{\frac{1}{3}} - k_1 [x_1]^{\frac{1}{3}} - k_2 [\hat{x}_2]^{\frac{1}{2}} \\ \dot{z} &= -k_3 \left[ x_1 + k_4 \hat{x}_2^{\frac{3}{2}} \right]^0\end{aligned}\tag{6}$$

which is a discontinuous system, whose trajectories are defined in the sense of Filippov [9]. If we introduce the estimation errors  $e_1 = \hat{x}_1 - x_1$ ,  $e_2 = \hat{x}_2 - x_2$  and the effect of the perturbation  $x_3 = z + \rho(t)$ , then the dynamics of the system can be rewritten as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 [x_1]^{\frac{1}{3}} - k_2 [\hat{x}_2]^{\frac{1}{2}} + x_3, \\ \dot{e}_1 &= -l_1 [e_1]^{\frac{2}{3}} + e_2 \\ \dot{e}_2 &= -l_2 [e_1]^{\frac{1}{3}} - x_3(t) \\ \dot{x}_3 &= -k_3 \left[ x_1 + k_4 (x_2 + e_2)^{\frac{3}{2}} \right]^0 + \dot{\rho}(t)\end{aligned}\tag{7}$$

We will prove Theorem 2 by showing that  $(x_1, x_2, e_1, e_2, x_3) = 0$ , which is an equilibrium point, is Globally Finite Time Stable. An important property of the previous systems is the homogeneity, that we recall briefly.

For a given vector  $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ , the dilation operator is defined as  $\Delta_\epsilon^r x := (\epsilon^{r_1} x_1, \dots, \epsilon^{r_n} x_n)$ ,  $\forall \epsilon > 0$ , where  $r_i > 0$  are the weights of the coordinates. Let  $\mathbf{r} = (r_1, \dots, r_n)$  be the vector of weights. A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  (respectively, a vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , or a vector-set field  $F(x) \subset \mathbb{R}^n$ ) is called  $\mathbf{r}$ -homogeneous of degree  $m \in \mathbb{R}$  if the identity  $V(\Delta_\epsilon^r x) = \epsilon^m V(x)$  holds (resp.,  $f(\Delta_\epsilon^r x) = \epsilon^l \Delta_\epsilon^r f(x)$ , or  $F(\Delta_\epsilon^r x) = \epsilon^l \Delta_\epsilon^r F(x)$ ), [8], [13]. Consider that the vector  $\mathbf{r}$  and dilation  $\Delta_\epsilon^r x$  are fixed. The homogeneous norm is defined by  $\|x\|_{\mathbf{r},p} := \left( \sum_{i=1}^n |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}$ ,  $\forall x \in \mathbb{R}^n$ , for any  $p \geq 1$ . The set  $S = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{r},p} = 1\}$  is the corresponding unit sphere. Homogeneous systems have important properties as e.g. that local stability implies global stability and if the homogeneous degree is negative asymptotic stability implies finite time stability [8], [13]: Assume that the origin of a Filippov Differential Inclusion,  $\dot{x} \in F(x)$ , is strongly locally Asymptotic Stable and the vector-set field  $F$  is  $r$ -homogeneous of degree  $l < 0$ ; then,  $x = 0$  is strongly globally finite-time stable and the settling time is continuous at zero and locally bounded.

System (7) is homogeneous, with weights  $\mathbf{r} = (3, 2, 3, 2, 1)$  for the variables  $(x_1, x_2, e_1, e_2, x_3)$  and negative homogeneous degree  $l = -1$ . From homogeneity arguments [13], [7], [14] one expects that the controllers have precision of order  $|x_1| \leq \nu_1 \tau^3$  and  $|x_2| \leq \nu_2 \tau^2$ , where  $\tau$  is the discretization step and  $\nu_1$  and  $\nu_2$  are constants depending only on the gains of the algorithm. Moreover, it is easy to show that for the Lyapunov functions the following inequality is satisfied

$$\dot{V}(x) \leq -\kappa V^{\frac{4}{5}}(x),$$

from which finite time convergence can be deduced. With the value of  $\kappa$  it is possible to estimate the convergence time as

$$T(x_0) \leq \frac{5}{\kappa} V^{\frac{1}{5}}(x_0).$$

### III. LYAPUNOV FUNCTION FOR THE CLOSED LOOP SYSTEM

We show, by using homogeneous and smooth Lyapunov Functions, that Theorems 1 and 2 are valid.

#### A. Proof of Theorem 1

Consider the closed loop system of plant (3) with the controller (4), with the variable  $x_3 = z + \rho(t)$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 [x_1]^{\frac{1}{3}} - k_2 [x_2]^{\frac{1}{2}} + x_3 \\ \dot{x}_3 &= -k_3 \left[ x_1 + k_4 x_2^{\frac{3}{2}} \right]^0 + \dot{\rho}(t).\end{aligned}\tag{8}$$

Consider the homogeneous and smooth Lyapunov Function

$$V(x_1, x_2, x_3) = \gamma_1 |\xi_1|^{\frac{5}{3}} + \gamma_{12} \xi_1 x_2 + |x_2|^{\frac{5}{2}} + \frac{1}{5} |x_3|^5,$$

where  $\xi_1 = x_1 - \frac{1}{k_1^3} \lceil x_3 \rceil^3$ . We recall Young's inequality.

*Lemma 3:* [10] For any positive real numbers  $a > 0$ ,  $b > 0$ ,  $c > 0$ ,  $p > 1$  and  $q > 1$ , with  $\frac{1}{p} + \frac{1}{q} = 1$ , the following inequality is always satisfied

$$ab \leq c^p \frac{a^p}{p} + c^{-q} \frac{b^q}{q}. \quad \blacksquare$$

From Lemma 3 it follows easily that given  $\gamma_{12}$  it is always possible to render  $V$  positive definite selecting  $\gamma_1$  sufficiently large.

Its derivative along the trajectories of (8) is given by

$$\dot{V} = W_1(\xi_1, x_2) + W_2(\xi_1, x_2, x_3) + W_3(x, \dot{\rho}),$$

where

$$\begin{aligned} W_1 &= \left( \frac{5}{3} \gamma_1 \lceil \xi_1 \rceil^{\frac{2}{3}} + \gamma_{12} x_2 \right) x_2 - \frac{5}{2} k_2 \left( \frac{2}{5} \gamma_{12} \xi_1 + \lceil x_2 \rceil^{\frac{3}{2}} \right) \left( \frac{k_1}{k_2} \lceil \xi_1 \rceil^{\frac{1}{3}} + \lceil x_2 \rceil^{\frac{1}{2}} \right) \\ W_2 &= -k_1 \left( \gamma_{12} \xi_1 + \frac{5}{2} \lceil x_2 \rceil^{\frac{3}{2}} \right) \alpha(\xi_1, x_3), \\ \alpha(\xi_1, x_3) &= \left[ \xi_1 + \frac{1}{k_1^3} \lceil x_3 \rceil^3 \right]^{\frac{1}{3}} - \left[ \frac{1}{k_1^3} \lceil x_3 \rceil^3 \right]^{\frac{1}{3}} - \lceil \xi_1 \rceil^{\frac{1}{3}}, \\ W_3(x, \dot{\rho}) &= \left( k_3 \left[ x_1 + k_4 x_2^{\frac{3}{2}} \right]^0 - \dot{\rho}(t) \right) |x_3|^2 \left[ \frac{3}{k_1^3} \left( \frac{5}{3} \gamma_1 \lceil \xi_1 \rceil^{\frac{2}{3}} + \gamma_{12} x_2 \right) - \lceil x_3 \rceil^2 \right]. \end{aligned}$$

Consider first  $W_1$ . If we set

$$\gamma_{12} = \frac{5}{2} \left( \frac{k_1}{k_2} \right)^3$$

then we get

$$\begin{aligned} W_1 &= \frac{5}{2} \left( \frac{k_1}{k_2} \right)^3 \left( \frac{2}{3} \left( \frac{k_2}{k_1} \right)^3 \gamma_1 \lceil \xi_1 \rceil^{\frac{2}{3}} + x_2 \right) x_2 - \\ &\quad \frac{5}{2} k_2 \left( \left( \frac{k_1}{k_2} \right)^3 \xi_1 + \lceil x_2 \rceil^{\frac{3}{2}} \right) \left( \frac{k_1}{k_2} \lceil \xi_1 \rceil^{\frac{1}{3}} + \lceil x_2 \rceil^{\frac{1}{2}} \right) \end{aligned}$$

The second term is negative semidefinite in  $(\xi_1, x_2)$ , and it becomes zero only at the set  $\mathcal{S}_1 = \left\{ x_2 = - \left( \frac{k_1}{k_2} \right)^2 \left[ x_1 - \frac{1}{k_1^3} \lceil x_3 \rceil^3 \right]^{\frac{2}{3}} \right\}$ .

On  $\mathcal{S}_1$  the value of  $W_1$  becomes

$$W_1|_{\mathcal{S}_1} = -\frac{5}{2} \left( \frac{k_1}{k_2} \right)^5 \left( \frac{2}{3} \left( \frac{k_2}{k_1} \right)^3 \gamma_1 - \left( \frac{k_1}{k_2} \right)^2 \right) |\xi_1|^{\frac{4}{3}},$$

which is negative if

$$\gamma_1 > \frac{3}{2} \left( \frac{k_1}{k_2} \right)^5,$$

i.e.  $\gamma_1$  is set sufficiently large. We recall the following well-known property of homogeneous functions

*Lemma 4:* [18], [19] Let  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be two homogeneous functions, with weights  $\mathbf{r} = (r_1, \dots, r_n)$  and degrees  $m$ , such that the following holds

$$\{x \in \mathbb{R}^n \setminus \{0\} : \gamma(x) = 0\} \subseteq \{x \in \mathbb{R}^n \setminus \{0\} : \eta(x) < 0\},$$

Then, there exists a real number  $\lambda^*$  such that, for all  $\lambda \leq \lambda^*$  for all  $x \in \mathbb{R}^n \setminus \{0\}$ , and some  $c > 0$ ,  $\eta(x) - \lambda \gamma(x) < -c \|x\|_{\mathbf{r}, p}^m$ .

■

Using Lemma 4 it follows that  $W_1 < -c \|(\xi_1, x_2)\|_{\mathbf{r}, p}^4$  for  $k_2$  sufficiently large.

Now we consider the continuous function, appearing in the third term

$$\alpha(\xi_1, x_3) = \left[ \xi_1 + \frac{1}{k_1^3} \lceil x_3 \rceil^3 \right]^{\frac{1}{3}} - \left[ \frac{1}{k_1^3} \lceil x_3 \rceil^3 \right]^{\frac{1}{3}} - \lceil \xi_1 \rceil^{\frac{1}{3}},$$

which is homogeneous of degree 1. Note that  $\alpha(\xi_1, 0) = 0$  and also  $\alpha(0, x_3) = 0$ . Since  $\alpha(\xi_1, x_3)$  is homogeneous in  $\xi_1$  it follows that

$$|\alpha(\xi_1, x_3)| \leq \delta(x_3) |\xi_1|^{\frac{1}{3}},$$

where  $\delta(x_3) \geq 0$  is continuous in  $x_3$  and  $\delta(0) = 0$ . This implies that, for some  $\beta > 0$

$$|W_2| \leq \delta(x_3) \beta \|(\xi_1, x_2)\|_{\mathbf{r}, p}^4,$$

and therefore, for small values of  $x_3$

$$\begin{aligned} W_1 + W_2 &\leq -(c - \beta \delta(x_3)) \|(\xi_1, x_2)\|_{\mathbf{r}, p}^4 \\ &\leq -c_2 \|(\xi_1, x_2)\|_{\mathbf{r}, p}^4. \end{aligned}$$

Due to the homogeneity of  $W_1 + W_2$  this must be valid globally. Note that  $W_1(\xi_1, x_2) + W_2(\xi_1, x_2, x_3) = 0$  on the set  $\mathcal{S}_2 = \{(\xi_1, x_2) = 0\}$ . Finally, the value of  $W_3$  on the set  $\mathcal{S}_2$  is given by

$$W_3|_{\mathcal{S}_2} \leq -(k_3 - L)|x_3|^4,$$

which is negative for  $L < k_3$ . Again, Lemma 4 implies that  $\dot{V} < 0$  selecting  $k_3$  (and  $L$ ) sufficiently small.

### B. Proof of Theorem 2

We first prove that the closed loop system, without perturbation and without integral controller, that we can write as (see (7))

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lceil x_1 \rceil^{\frac{1}{3}} - k_2 \lceil x_2 \rceil^{\frac{1}{2}} + \omega(x_2, e_2) \\ \dot{e}_1 &= -l_1 \lceil e_1 \rceil^{\frac{2}{3}} + e_2 \\ \dot{e}_2 &= -l_2 \lceil e_1 \rceil^{\frac{1}{3}} \\ \omega(x_2, e_2) &= k_2 \lceil x_2 \rceil^{\frac{1}{2}} - k_2 \lceil x_2 + e_2 \rceil^{\frac{1}{2}}, \end{aligned} \tag{9}$$

is globally finite time stable.

We use the homogeneous and smooth LF

$$V(x, e) = V_1(x) + \mu V_2(e),$$

where  $\mu > 0$ ,

$$V_1(x) = \gamma_1 |x_1|^{\frac{5}{3}} + \frac{5}{2} \left( \frac{k_1}{k_2} \right)^3 x_1 x_2 + |x_2|^{\frac{5}{2}},$$

$$V_2(e) = |\epsilon_1|^{\frac{5}{3}} + \gamma_2 |e_2|^{\frac{5}{2}},$$

and  $\epsilon_1 = e_1 - \frac{1}{l_1^{\frac{2}{3}}} \lceil e_2 \rceil^{\frac{3}{2}}$ .  $V_2$  is obviously p.d. for  $\gamma_2 > 0$ , while for  $V_1$  it follows from Young's inequality (as in the previous subsection) that it is positive definite selecting  $\gamma_1$  sufficiently large. The derivative of  $V_2$  is

$$\dot{V}_2(e) = -\frac{5}{3} l_1 \left[ e_1 - \frac{1}{l_1^{\frac{2}{3}}} \lceil e_2 \rceil^{\frac{3}{2}} \right]^{\frac{2}{3}} \left( \lceil e_1 \rceil^{\frac{2}{3}} - \frac{1}{l_1} e_2 \right) - \frac{5}{2} l_2 \gamma_2 \lceil e_1 \rceil^{\frac{1}{3}} \lceil e_2 \rceil^{\frac{3}{2}}.$$

The first term is negative, except at the set  $\mathcal{S}_3 = \left\{ e_1 = \frac{1}{l_1^{\frac{2}{3}}} \lceil e_2 \rceil^{\frac{3}{2}} \right\}$ , on which the value of  $\dot{V}_2$  is

$$\dot{V}_2|_{\mathcal{S}_3} = -\frac{5}{2} \frac{l_2}{l_1^{\frac{1}{2}}} \gamma_2 |e_2|^2 < 0.$$

And therefore, due to Lemma 4, it is possible to render  $\dot{V}_2 < 0$  selecting  $\gamma_2$  sufficiently small. The derivative of  $V_1$  along the trajectories of (9) is given by

$$\begin{aligned} \dot{V}_1(x) &= \left( \frac{5}{3} \gamma_1 \lceil x_1 \rceil^{\frac{2}{3}} + \frac{5}{2} \left( \frac{k_1}{k_2} \right)^3 x_2 \right) x_2 - \frac{5}{2} k_2 \left( \left( \frac{k_1}{k_2} \right)^3 x_1 + \lceil x_2 \rceil^{\frac{3}{2}} \right) \left( \frac{k_1}{k_2} \lceil x_1 \rceil^{\frac{1}{3}} + \lceil x_2 \rceil^{\frac{1}{2}} \right) \\ &\quad + \frac{5}{2} k_2 \left( \left( \frac{k_1}{k_2} \right)^3 x_1 + \lceil x_2 \rceil^{\frac{3}{2}} \right) \omega(x_2, e_2). \end{aligned}$$

The second term is negative, except at the set  $\left\{ x_1 = -\left( \frac{k_2}{k_1} \right)^3 \lceil x_2 \rceil^{\frac{3}{2}} \right\}$ , at which the value of  $\dot{V}_1$  is

$$\dot{V}_1(x_1, x_2) = -\left( \frac{5}{3} \left( \frac{k_2}{k_1} \right)^2 \gamma_1 - \frac{5}{2} \left( \frac{k_1}{k_2} \right)^3 \right) |x_2|^2,$$

that is negative if  $\gamma_1$  is sufficiently large, i.e.

$$\gamma_1 > \frac{3}{2} \left( \frac{k_1}{k_2} \right)^5.$$

Thanks to Lemma 4 the first two terms in  $\dot{V}$  can be made n.d. selecting  $k_2$  sufficiently large. Note furthermore that the function  $[x]^{\frac{1}{2}}$  is Hölder continuous, and therefore

$$|\omega(x_2, e_2)| \leq ck_2 |e_2|^{\frac{1}{2}}$$

everywhere, for some  $c > 0$ . We obtain therefore

$$\dot{V} \leq -\alpha_1 \|x\|_{r,p}^4 + \alpha_2 ck_2 \|x\|_{r,p}^3 \|e\|_{r,p} - \mu \alpha_3 \|e\|_{r,p}^4,$$

for some positive  $\alpha_i$ . Selecting  $\mu$  sufficiently large we obtain  $\dot{V} < 0$ . Q.E.D.

We notice that system (9) is not insensitive to the perturbation  $\rho(t)$ . However, as in the previous case adding the discontinuous integral controller a Lipschitz continuous perturbation can be completely compensated. The proof is similar to that case.

#### IV. SIMULATION EXAMPLE

We illustrate the behavior of the proposed integral controllers by some simulations. Consider the dynamics of a simple pendulum without friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l} \sin(x_1) + \frac{1}{ml^2} u + \rho(t), \end{aligned}$$

where  $x_1 = \theta$  is the position angle,  $x_2 = \dot{\theta}$  is the angular velocity,  $m$  is the mass of the bob,  $g$  is the gravity acceleration,  $l$  is the length of the bob, the control  $u$  is the torque applied to the pendulum, and the perturbation  $\rho(t) = 0.4 \sin(t)$ , that can be interpreted also as the second derivative of a reference signal (in this case the state  $x$  corresponds to the tracking error). For the simulations we have used the following parameter values  $l = 1$  [m],  $m = 1.1$  [Kg],  $g = 9.815$  [m/s<sup>2</sup>], and the initial conditions  $x_1(0) = 2$ ,  $x_2(0) = 2$ .

We have implemented three controllers:

- A State Feedback (SF) controller with discontinuous integral term, as given by (4), with gains  $k_1 = 2$ ,  $k_2 = 5$ ,  $k_3 = 0.5$ ,  $k_4 = 0$ , and initial value of the integrator  $z(0) = 0$ .
- An Output Feedback (OF) controller with discontinuous integral term, as given by (5), with controller gains  $k_1 = 2\lambda^{\frac{2}{3}}$ ,  $k_2 = 5\lambda^{\frac{1}{2}}$ ,  $k_3 = 0.5\lambda$ ,  $k_4 = 0$ ,  $\lambda = 3$ , observer gains  $l_1 = 2L$ ,  $l_2 = 1.1L^2$ ,  $L = 4$ , observer initial conditions  $\hat{x}_1(0) = 0$ ,  $\hat{x}_2(0) = 0$ , and initial value of the integrator  $z(0) = 0$ .
- A Twisting controller [17], [7], [14], given by  $u = -k_1 [x_1]^0 - k_2 [x_2]^0$ , with gains  $k_1 = 1.2$ ,  $k_2 = 0.6$ .

The simulations for the three controllers are presented in Figures IV-4. In Figure IV the evolution of the position is presented and also the evolution of the estimated position given by the observer for the OF, which converges very fast. All controllers are able to bring the position to zero in finite time.

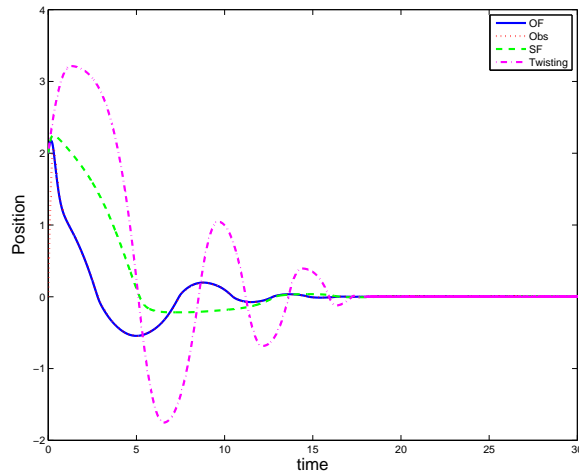


Fig. 1. Behavior of  $x_1$  with Twisting and the Discontinuous Integral Controller

Figure IV presents the time evolution of the velocity and its estimation by the observer for the OF, which converges in finite time around the time 15. We see also the typical zig-zag behavior for the Twisting controller. All controllers are able to bring the velocity to rest in finite time.

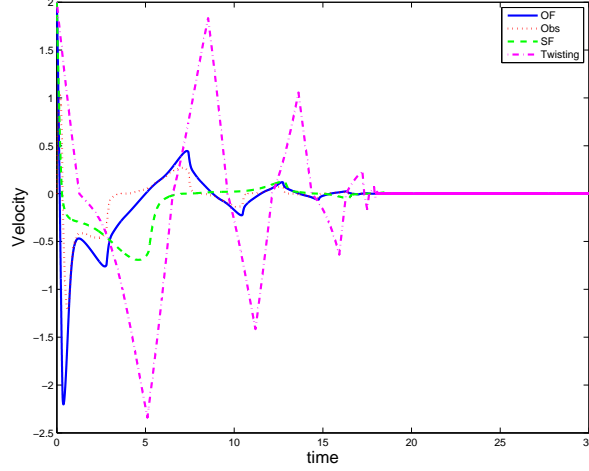


Fig. 2. Behavior of  $x_2$  with Twisting and the Discontinuous Integral Controller

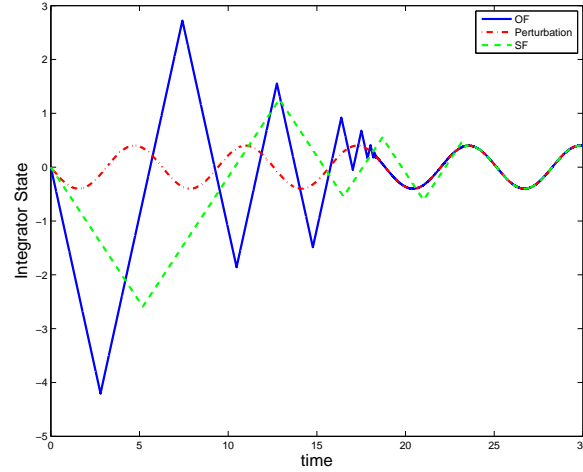


Fig. 3. Behavior of the state of the discontinuous Integrator and the negative value of the perturbation  $-\rho(t)$ .

In Figure 3 the integrator state is presented for both controllers OF and SF, and the (negative) value of the perturbation ( $-\rho(t)$ ). We note the zig-zag behavior of the integral controller, due precisely to its discontinuous character. We appreciate also that the integrator signal reconstructs after a finite time the (negative value of the) perturbation, and this is the reason for it to be able to fully compensate its action on the plant.

Finally, Figure 4 presents the control signal  $u$  for the three controllers. We see that, while the OF and the SF controllers with discontinuous Integral action provide a continuous control signal, the Twisting controller provides a switching (discontinuous) control signal, with an extremely high frequency when the equilibrium has been reached, which corresponds to the (undesirable) chattering phenomenon.

## V. CONCLUSION

We present in this paper a Discontinuous Integral Controller, which shares the properties of the classical PID control and the HOSM controllers: Similar to HOSM it is able to fully compensate a Lipschitz perturbation or to track an (unknown) time varying reference with bounded second derivative, it has high precision due to the homogeneity properties, and it stabilizes the origin globally and in finite time. Similar to the PID control it has a continuous control signal. In order to achieve an Output Feedback scheme we introduce a finite time converging observer. The stability proofs are performed with a novel Lyapunov method. It is possible to extend this idea to systems with higher relative degree, and this will be done in future work.

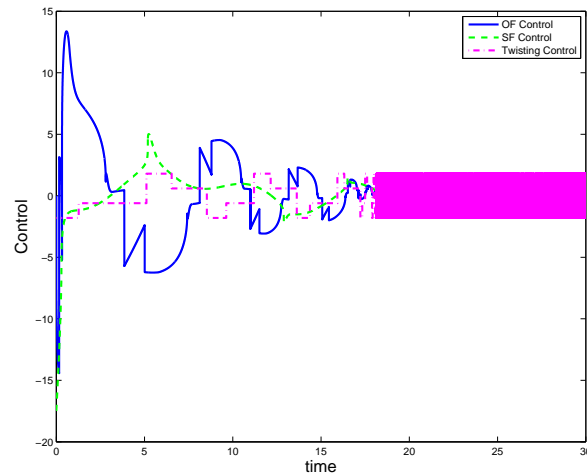


Fig. 4. Control signal for OF and SF controllers with discontinuous Integral action and for the Twisting controller.

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